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HYBRID ELEMENT METHOD FOR INCOMPRESSIBLE AND NEARLY INCOMPRESSIBLE MATERIALS

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Abstract—A new hybrid element method suitable for problems with different Poisson's ratios, including incompressible and nearly incompressible materials, is proposed. This hybrid model does not exhibit any locking phenomenon for nearly incompressible materials, and is capable of producing correct displacement and stress solutions in the case of uniform stress and zero displacement state for incompressible materials. In addition, the model has other favourable characteristics such as having no extra zero energy modes, being coordinate invariant, possessing high accuracy and requiring simple manipulations in the formulation. A new variational functional suitable for different Poisson's ratios is proposed here. This functional is given in terms of a number of independent variables which include two stresses, one strain, two displacements and two average compressive stresses. A plane strain quadrilateral element Q_{4-LL} can be established based on the proposed hybrid element method. Through a series of worked examples it is demonstrated that the element can be used for various Poisson's ratios, possesses high accuracy and will not exhibit locking. By comparing the proposed element with existing elements for incompressible and nearly incompressible materials, it is possible to determine their relationship and to establish the fact that the hybrid method is a unified method incorporating many displacement models.

INTRODUCTION

When standard conforming displacement elements are used for problems involving nearly incompressible materials, the locking phenomenon will become increasingly apparent as the Poisson's ratio approaches the value of 0.5, and the problem will become singular for the case of incompressible material. In general, two methods of solution are available. The first one is directed towards single field displacement models, and requires the imposition of certain constraints. The other method deals with multiple field models, in which constraint conditions suitable for various Poisson's ratios are introduced.

A functional suitable for incompressible and nearly incompressible material problems was first proposed by Herrmann and Toms (1964), and Herrmann (1965), such a functional included an average compressive stress variable and was applied to finite element analysis successfully by Fried (1974), and Kuai and Liu (1983). The functional was subsequently proven to be a special case of the more general Reissner principle by Key (1969).

Since both displacements and average compressive stresses are taken as nodal variables, the finite elements developed are in fact mixed formulation models, which may yield stiffness matrices that are not positive definite and are thus, difficult to solve. A more satisfactory alternative method, at least for nearly incompressible materials, is to regard the average compressive stresses as independent internal parameters which are subsequently eliminated through static condensation. However, for the case of incompressible materials, it would be necessary to employ special element elimination techniques (such as the special front solution proposed by Kuai and Liu (1983)) in order to obtain solutions. While it is true that the above-mentioned finite element models will not exhibit the locking phenomenon and also provide a unified method for problems involving different Poisson's ratios, they nevertheless suffer from the disadvantage of not being able to provide accurate solutions, particularly for the case of simple, lower-order elements.

The use of reduced integration in single field conforming displacement elements for the computation of nearly incompressible material problems was suggested by Hughes (1977, 1980). This simple technique can also alleviate locking, although it cannot be applied directly to incompressible material problems. A single field conforming displacement element model with generalized displacements was introduced by Zhong and Lee (1982), and Li and Liu (1986). By introducing incompressible constraint conditions into the element formulation, it is possible to solve the singularity problem connected with incompressible materials. It should be noted that such a procedure is really equivalent to a special type of condensation procedure.

In comparison with the single field conforming displacement element models mentioned above, multiple field hybrid elements can in general offer higher accuracy, and in many cases can be used to deal with nearly incompressible material problems without making special efforts. Several hybrid elements were developed by Pian and Lee (1976) and by Spilker and Pian (1978) and Spilker (1981) for axisymmetrical bodies with nearly incompressible materials.

Recently, a new approach called the quasi-conforming element technique for the penalty finite element method was presented by Tang and Liu (1985). In this technique the dilatational strain and the deviatoric strain are both regarded as independent variables, while the term in the $1/(1-2\nu)$ strain energy expression is treated as a penalty factor as $\nu \to 0.5$. The adoption of the quasi-conforming element method at the element level will produce extra zero energy in the terms with $1/(1-2\nu)$, which in turn will guarantee non-trivial solutions as $\nu \to 0.5$, thus successfully overcoming the locking problem for nearly incompressible material problems. The rectangular element Q_{4-L} by Tang and Liu (1985) based on this technique has been found to be equivalent to the rectangular element developed by Hughes based on reduced integration.

From the above discussions it can be seen that the full potential of the multiple field hybrid method and the quasi-conforming element method has yet to be explored. It seems entirely possible to be able to develop a hybrid model with high accuracy and high computational efficiency, which at the same time can deal with materials with various Poisson's ratios (including incompressible and nearly incompressible materials).

VARIATIONAL PRINCIPLE AND HYBRID MODEL FOR INCOMPRESSIBLE AND NEARLY INCOMPRESSIBLE MATERIALS

Variational principle

In order to reflect the variational constraint condition of zero (or approaching zero) dilatational strain for incompressible or nearly incompressible materials, it would be appropriate to write the strain energy due to the dilatational strain separately from the other terms of the energy expression. To this end, the minimum total potential energy principle should be rewritten as

$$\pi_p(\mathbf{u}_1) = \iiint_V \left(\tfrac{1}{2}\varepsilon_1^T A_1 \varepsilon_1 + \tfrac{1}{2}\lambda\theta^2 - \bar{\mathbf{f}}^T \mathbf{u}_1\right) dv - \iint_{S_T} \bar{\mathbf{T}}^T \mathbf{u}_1 \, ds \tag{1}$$

in which the strains

$$\varepsilon_1 = \mathbf{D}^T \mathbf{u}_1 = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

and $\mathbf{u}_1 = [u, v, w]^T$ are the displacement vectors, $\bar{\mathbf{f}}$ and $\bar{\mathbf{T}}$ are the body forces in V and the boundary forces on S_T, respectively. The elasticity coefficient λ and the elasticity matrix $\mathbf{A}_1$ are

$$\lambda = \frac{Ev}{(1+v)(1-2v)}$$

$$\mathbf{A}_1 = 2G\begin{bmatrix} 1 &&&&& \\ & 1 &&&& \\ && 1 &&& \\ &&& \frac{1}{2} && \\ &&&& \frac{1}{2} & \\ &&&&& \frac{1}{2} \end{bmatrix}$$

where E, v, and G, are Young's modulus, Poisson's ratio, and shear modulus, respectively ($G = E/2(1+v)$).

It is obvious that as $v \to 0.5$, $\lambda\theta^2$ will approach infinity unless θ also approaches zero, thus resulting in the locking phenomenon.

The concept of average compressive stress ($H = \theta/(1-2v)$), which was introduced by Herrmann (1965), can be incorporated into π_p through the use of Lagrange multipliers, i.e.

$$\pi_R^*(\mathbf{u}_1, H, \alpha) = \iiint_V \left[\tfrac{1}{2}\boldsymbol{\varepsilon}_1^T \mathbf{A}_1 \boldsymbol{\varepsilon}_1 + Gv(1-2v)H^2 - \bar{\mathbf{f}}^T\mathbf{u}_1 + \alpha\left(H - \frac{\theta}{1-2v}\right)\right] dv - \iint_{S_T} \bar{\mathbf{T}}^T\mathbf{u}_1 \, ds. \tag{2}$$

The Lagrange multiplier α can be obtained from $\delta\pi_R^* = 0$, from which

$$\alpha = -2Gv(1-2v)H. \tag{3}$$

Substituting eqn (3) into eqn (2) yields

$$\pi_R(\mathbf{u}_1, H) = \iiint_V (\tfrac{1}{2}\boldsymbol{\varepsilon}_1^T\mathbf{A}_1\boldsymbol{\varepsilon}_1 - G(1-2v)vH^2 + 2GvH\theta - \bar{\mathbf{f}}^T\mathbf{u}_1)\, dv - \iint_{S_T} \bar{\mathbf{T}}^T\mathbf{u}_1 ds. \tag{4}$$

Equation (4) is in fact the functional established by Herrmann.

If the method proposed by Chen and Cheung (1987a,b) is adopted, then new variables must be introduced into π_R. Ignoring terms due to external forces for the time being, the new functional of each element V_e for deriving the element stiffness matrix can be written as

$$\pi_G^e(\varepsilon, \sigma, \sigma_1, u, u_\lambda) = \iiint_{V_e} [\tfrac{1}{2}\boldsymbol{\varepsilon}^T\mathbf{A}\boldsymbol{\varepsilon} - \boldsymbol{\sigma}^T(\boldsymbol{\varepsilon} - \mathbf{D}_1^T\mathbf{u}) - (\mathbf{D}_0\boldsymbol{\sigma})^T\mathbf{u}_\lambda + \mu(\boldsymbol{\sigma}_1 - \mathbf{A}\boldsymbol{\varepsilon})^T(\boldsymbol{\varepsilon} - \mathbf{a}\boldsymbol{\sigma}_1)]\, dv \tag{5a}$$

in which the generalized strain is given by

$$\boldsymbol{\varepsilon} = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}, H_e]^T$$

where H_e is the average compressive stress of the element. The generalized stress is given by

$$\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}, H_0]^T$$

$$\boldsymbol{\sigma}_1 = [\sigma'_x, \sigma'_y, \sigma'_z, \tau'_{yz}, \tau_{xz}, \tau_{xy}, H'_0]^T$$

where H_0 is the multiplier for $H_e - H = 0$. The generalized displacement is given by

$$\mathbf{u} = [\mathbf{u}_q, H]^T$$

where $\mathbf{u}_q = [u_q, v_q, w_q]^T$ is the displacement expressed in terms of nodal parameters, H the average compressive stress expressed in terms of element internal parameters. The displacement expressed in terms of the internal parameters is given by

$$\mathbf{u}_\lambda = [u_\lambda, v_\lambda, w_\lambda]^T$$

where $\mathbf{A}$ and $\mathbf{a}$ are the elasticity matrices, with $\mathbf{A} = \mathbf{a}^{-1}$ and μ is any prescribed constant. The various matrices can be written in explicit form as follows:

$$\mathbf{A} = 2G \begin{bmatrix} 1 & & & & & & \nu \\ & 1 & & & & & \nu \\ & & 1 & & & & \nu \\ & & & \frac{1}{2} & & & 0 \\ & & & & \frac{1}{2} & & 0 \\ & & & & & \frac{1}{2} & 0 \\ \nu & \nu & \nu & 0 & 0 & 0 & -\nu(1-2\nu) \end{bmatrix} \tag{5b}$$

$$\mathbf{a} = \frac{1}{2G} \begin{bmatrix} 1-\frac{\nu}{1+\nu} & -\frac{\nu}{1+\nu} & -\frac{\nu}{1+\nu} & 0 & 0 & 0 & \frac{1}{1+\nu} \\ -\frac{\nu}{1+\nu} & 1-\frac{\nu}{1+\nu} & -\frac{\nu}{1+\nu} & 0 & 0 & 0 & \frac{1}{1+\nu} \\ -\frac{\nu}{1+\nu} & -\frac{\nu}{1+\nu} & 1-\frac{\nu}{1+\nu} & 0 & 0 & 0 & \frac{1}{1+\nu} \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ \frac{1}{1+\nu} & \frac{1}{1+\nu} & \frac{1}{1+\nu} & 0 & 0 & 0 & -\frac{1}{\nu(1+\nu)} \end{bmatrix} \tag{5c}$$

$$\mathbf{D}_0 = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \end{bmatrix} \tag{5d}$$

$$\mathbf{D}_1^T = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{5e}$$

As usual, the basic equations for the element can be obtained by taking the variational equation $\delta\pi_G^e = 0$

$$\delta\pi_G^e = \iiint_{V_e} \{\delta\boldsymbol{\varepsilon}^T[(\mathbf{A}\boldsymbol{\varepsilon}-\boldsymbol{\sigma})+2\mu(\boldsymbol{\sigma}_1-\mathbf{A}\boldsymbol{\varepsilon})]-\delta\boldsymbol{\sigma}^T(\boldsymbol{\varepsilon}-\mathbf{D}_1^T\mathbf{u}-\mathbf{D}_0^T\mathbf{u}_\lambda)+\delta\boldsymbol{\sigma}_1^T(2\boldsymbol{\varepsilon}-2\mathbf{a}\boldsymbol{\sigma}_1)$$

$$+\delta(\mathbf{u}_q+\mathbf{u}_\lambda)^T(\mathbf{D}_0\boldsymbol{\sigma})+\delta H\cdot H_0\}\,\mathrm{d}v+\iint_{\partial V_e}(-\delta\mathbf{T}^T\mathbf{u}_\lambda+\mathbf{T}^T\delta\mathbf{u}_q)\,\mathrm{d}s = 0.$$

The Euler equation is equivalent to

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbf{A}\boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} &= \mathbf{D}_1^T\mathbf{u}+\mathbf{D}_0^T\mathbf{u}_\lambda \\ \mathbf{D}_0\boldsymbol{\sigma} &= \mathbf{0} \\ H_0 &= 0 \\ \boldsymbol{\sigma} &= \boldsymbol{\sigma}_1. \end{aligned} \tag{6}$$

The equations $(1-2\nu)H = \theta$, $H_c = H$, and $H'_0 = 0$ are included in eqns (6). It is obvious that when $\nu = 1/2$, the incompressible condition of $\theta = 0$ is arrived at. Compatibility of interelement traction $\mathbf{T}$ and interelement displacement $\mathbf{U}_q+\mathbf{U}_\lambda$ will be obtained by term

$$\iint_{\partial V_e}(\ldots)\,\mathrm{d}s = 0$$

in $\delta\pi_G^e = 0$.

Hybrid element models

All the variables $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}_1$, $\boldsymbol{\varepsilon}$, $\mathbf{u}_\lambda$, $\mathbf{u}$, etc. in π_G are independent of each other. Since eqns (5) have similar forms to the functional by Chen and Cheung (1987a,b), they can be easily used as a basis for the formulation of the finite element. Isoparametric interpolation functions appear to be most suitable for the purpose of developing hybrid elements, since the resulting element can be guaranteed to be coordinate invariant, and to have a simple formulation and relatively high accuracy.

If the variables are given as

$$\begin{aligned} \boldsymbol{\varepsilon} &= \frac{1}{|J|}\mathbf{N}\boldsymbol{\alpha} \\ \boldsymbol{\sigma} &= \mathbf{p}\boldsymbol{\beta} \\ \boldsymbol{\sigma}_1 &= \mathbf{p}_1\boldsymbol{\beta}_1 \\ \mathbf{u} &= \mathbf{F}\mathbf{q} \\ \mathbf{u}_\lambda &= \mathbf{M}\boldsymbol{\lambda} \end{aligned} \tag{7}$$

then the stiffness matrix can be written as

$$K^e = \mathbf{G}^T\mathbf{W}^{-T}\mathbf{H}\mathbf{W}^{-1}\mathbf{G} \tag{8}$$

and the stress vector as

$$\boldsymbol{\sigma}_1 = \mathbf{P}_1\mathbf{W}_1^{-T}\mathbf{H}\mathbf{W}^{-1}\mathbf{G}\mathbf{q} \tag{9}$$

in which

$$\mathbf{H} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{1}{|J|}\mathbf{N}^{T}\mathbf{A}\mathbf{N}\,d\zeta\,d\eta\,d\tau$$

$$\mathbf{W} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{P}^{T}\mathbf{N}\,d\zeta\,d\eta\,d\tau$$

$$\mathbf{G} = [\mathbf{G}_1, -\mathbf{G}_2]$$

$$\mathbf{G}_1 = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{P}^{T}(\mathbf{D}_1^{T}\mathbf{F})\,|J|\,d\zeta\,d\eta\,d\tau$$

$$\mathbf{G}_2 = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (\mathbf{D}_0\mathbf{P})^{T}\mathbf{M}\,|J|\,d\zeta\,d\eta\,d\tau$$

$$\mathbf{W}_1 = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{P}_1^{T}\mathbf{N}\,d\zeta\,d\eta\,d\tau$$

$$\mathbf{q} = [\mathbf{q}^{e}, \mathbf{h}^{e}, \boldsymbol{\lambda}]^{T}.$$

It is observed that $\mathbf{q}$ includes the nodal parameter $\mathbf{q}^e$, the element average compressive stress parameter $\mathbf{h}^e$, and the internal displacement parameter $\boldsymbol{\lambda}$ and is therefore known as the generalized nodal parameter. It should be noted that as a result of using isoparametric interpolations, both $\mathbf{W}$ and $\mathbf{W}_1$ are diagonal matrices.

Quadrilateral hybrid element for plane strain problems

A number of quadrilateral elements (Fig. 1) have already been established for the calculation of plane strain problems with nearly incompressible materials by Hughes (1980), Li and Liu (1986), and Tang and Liu (1985). The compatible displacement is well known and is given by

$$\mathbf{u}_q = \begin{Bmatrix} u \\ v \end{Bmatrix} = \sum_{i=1}^{4} \tfrac{1}{4}(1+\zeta\zeta_i)(1+\eta\eta_i)\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} \tag{10}$$

while the isoparametric transformation is

$$\begin{aligned} \begin{Bmatrix} x \\ y \end{Bmatrix} &= \sum_{i=1}^{4} (1+\zeta\zeta_i)(1+\eta\eta_i)\begin{Bmatrix} x_i \\ y_i \end{Bmatrix} \\ &= \begin{Bmatrix} a_1 + a_2\zeta + a_3\eta + a_4\zeta\eta \\ b_1 + b_2\zeta + b_3\eta + b_4\zeta\eta \end{Bmatrix} \end{aligned} \tag{11}$$

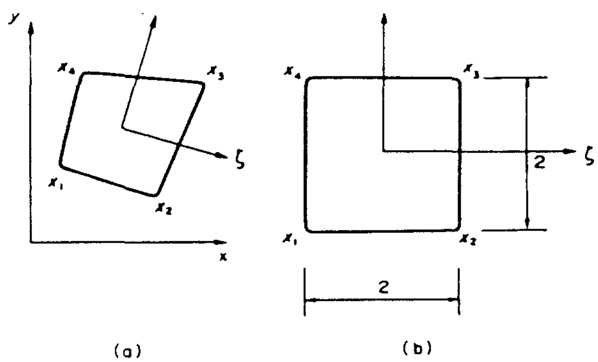

Fig. 1. A typical quadrilateral element.

in which u_i, v_i and x_i, y_i are the nodal displacements and nodal coordinates, respectively, and

$$\begin{aligned}
a_1 &= \tfrac{1}{4}(x_1+x_2+x_3+x_4)\\
a_2 &= \tfrac{1}{4}(-x_1+x_2+x_3-x_4)\\
a_3 &= \tfrac{1}{4}(-x_1-x_2+x_3+x_4)\\
a_4 &= \tfrac{1}{4}(x_1-x_2+x_3-x_4)\\
b_1 &= \tfrac{1}{4}(y_1+y_2+y_3+y_4)\\
b_2 &= \tfrac{1}{4}(-y_1+y_2+y_3-y_4)\\
b_3 &= \tfrac{1}{4}(-y_1-y_2+y_3+y_4)\\
b_4 &= \tfrac{1}{4}(y_1-y_2+y_3-y_4).
\end{aligned} \tag{12}$$

For the plane strain problem in question, eqns (5) have to be reduced to the corresponding two-dimensional form, in which case

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ H_\varepsilon \end{Bmatrix} = \frac{1}{|J|}\begin{bmatrix} \mathbf{N}_1 & & & \\ & \mathbf{N}_1 & & \\ & & \mathbf{N}_1 & \\ & & & \mathbf{N}_2 \end{bmatrix}\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{12} \end{Bmatrix} \tag{13a}$$

$$\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ H_0 \end{Bmatrix} = \begin{bmatrix} \mathbf{P}_1 & & & \\ & \mathbf{P}_1 & & \\ & & \mathbf{P}_1 & \\ & & & \mathbf{P}_2 \end{bmatrix}\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{12} \end{Bmatrix} \tag{13b}$$

$$\boldsymbol{\sigma}_1 = \begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ \tau'_{xy} \\ H'_0 \end{Bmatrix} = \begin{bmatrix} \mathbf{P}'_1 & & & \\ & \mathbf{P}'_1 & & \\ & & \mathbf{P}'_1 & \\ & & & \mathbf{P}'_2 \end{bmatrix}\begin{Bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{12} \end{Bmatrix} \tag{13c}$$

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_q \\ \mathbf{H} \end{Bmatrix} = [\mathbf{F} \quad \mathbf{N}_2]\begin{Bmatrix} \mathbf{q}^e \\ \mathbf{h}^e \end{Bmatrix} \tag{13d}$$

$$\mathbf{q}^e = [u_1, u_2, u_3, u_4 \quad v_1, v_2, v_3, v_4]^T$$

$$\mathbf{h}^e = [h_1, h_2, h_3]$$

$$\mathbf{u}_\lambda = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} M_1 & M_2 & & \\ & & M_1 & M_2 \end{bmatrix}\begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{Bmatrix}. \tag{13e}$$

Three different elements can be established through the choice of different interpolating functions which are listed in Table 1.

The various matrices required for making up the stiffness matrix K_e and the stress

Table 1. Interpolating functions for different elements

Function	Element Q_{4-LL}	Q_{4-C}	Q_{4-L}
$\mathbf{N}_1$	$1, \zeta, \eta$	$1, \zeta, \eta$	$1, \zeta, \eta$
$\mathbf{N}_2$	$\lvert J\rvert (1, \zeta, \eta)$	$\lvert J\rvert$	$\lvert J\rvert (1, \zeta, \eta)$
$\mathbf{P}_1$	$1, \zeta\eta^2, \zeta^2\eta$	$1, \zeta\eta^2, \zeta^2\eta$	$1, \zeta\eta^2, \zeta^2\eta$
$\mathbf{P}_2$	$\frac{1}{\lvert J\rvert}(1, \zeta, \eta)$	$\frac{1}{\lvert J\rvert}$	$\frac{1}{\lvert J\rvert}(1, \zeta, \eta)$
$\mathbf{P}'_1$	$1, \zeta, \eta$	$1, \zeta, \eta$	$1, \zeta, \eta$
$\mathbf{P}'_2$	$\frac{1}{\lvert J\rvert}(1, \zeta, \eta)$	$\frac{1}{\lvert J\rvert}$	$\frac{1}{\lvert J\rvert}(1, \zeta, \eta)$
$\mathbf{M}^1$	$\frac{1}{3}-\zeta^2$	0	0
$\mathbf{M}_2$	$\frac{1}{3}-\eta^2$	0	0
Remarks	Linear interpolation for strains Three average compressive stress parameters Four internal displacement parameters	Linear interpolation for strains One average compressive stress parameter No internal displacement parameters	Linear interpolation for strains Three average compressive stress parameters No internal displacement parameters

matrix have been worked out explicitly and they are given as follows:

$$\mathbf{A} = 2G \begin{bmatrix} 1 & & & \nu \\ & 1 & & \nu \\ & & \frac{1}{2} & 0 \\ \nu & \nu & 0 & -\nu(1-2\nu) \end{bmatrix}$$

$$\mathbf{H} = \bar{\mathbf{A}} \int_{-1}^{1} \int_{-1}^{1} \bar{\mathbf{H}} \frac{1}{|\mathbf{J}|} \, d\zeta \, d\eta$$

in which

$$\bar{\mathbf{H}} = \begin{bmatrix} \mathbf{H}_1 & & & \\ & \mathbf{H}_2 & & \\ & & \mathbf{H}_3 & \\ & & & \mathbf{H}_4 \end{bmatrix}, \quad H_i = \begin{cases} \mathbf{N}_1^T \mathbf{N}_1 & (i \leqslant 3) \\ \mathbf{N}_2^T \mathbf{N}_2 & (i = 4) \end{cases}$$

$$\bar{\mathbf{A}} = 2G \begin{bmatrix} \mathbf{I}_3 & & & \nu\mathbf{I}_m \\ & \mathbf{I}_3 & & \nu\mathbf{I}_m \\ & & \frac{1}{2}\mathbf{I}_3 & 0 \\ \nu\mathbf{I}_m & \nu\mathbf{I}_m & 0 & -\nu(1-2\nu)\mathbf{I}_m \end{bmatrix}$$

where $\mathbf{I}_3$ is a 3×3 unit matrix, $\mathbf{I}_m$ a $m \times m$ unit matrix ($m = 3$ for Q_{4-LL} and Q_{4-L}, and $m = 1$ for Q_{4-C}).

The diagonal matrices $\mathbf{W}^{-1}$ and $\mathbf{W}_1^{-1}$ for the three elements are given in Table 2.

Table 2. $\mathbf{W}^{-1}$ and $\mathbf{W}_1^{-1}$ matrices

	Q_{4-LL}, Q_{4-L}	Q_{4-C}
diag ($\mathbf{W}^{-1}$)	$\frac{1}{4}(1, 9, 9, 1, 9, 9, 1, 9, 9, 1, 3, 3)$	$\frac{1}{4}(1, 9, 9, 1, 9, 9, 1, 9, 9, 1)$
diag ($\mathbf{W}_1^{-1}$)	$\frac{1}{4}(1, 3, 3, 1, 3, 3, 1, 3, 3, 1, 3, 3)$	$\frac{1}{4}(1, 3, 3, 1, 3, 3, 1, 3, 3, 1)$

The matrices necessary for establishing $\mathbf{W}^{-1}\mathbf{G}$, for element Q_{4-LL} only, are given explicitly as

$$\mathbf{W}^{-1}\mathbf{G}_1 = \begin{bmatrix} \mathbf{A}_x & 0 & 0 \\ 0 & \mathbf{A}_y & 0 \\ \mathbf{A}_y & \mathbf{A}_x & 0 \\ 0 & 0 & \mathbf{I}_3 \end{bmatrix}$$

in which

$$\mathbf{A}_x = \frac{1}{4}\begin{bmatrix} b_2-b_3 & b_2+b_3 & -b_2+b_3 & -b_2-b_3 \\ -b_2-b_4 & b_2+b_4 & -b_2+b_4 & b_2-b_4 \\ b_4+b_3 & b_4-b_3 & -b_4+b_3 & -b_4-b_3 \end{bmatrix}$$

$$\mathbf{A}_y = \frac{1}{4}\begin{bmatrix} -a_2+a_3 & -a_2-a_3 & a_2-a_3 & a_2+a_3 \\ a_2+a_4 & -a_2-a_4 & a_2-a_4 & -a_2+a_4 \\ -a_2-a_3 & -a_4+a_3 & a_4-a_3 & a_4+a_3 \end{bmatrix}$$

and

$$-\mathbf{W}^{-1}\mathbf{G}_2 = \begin{bmatrix} \mathbf{B}_1 & 0 \\ 0 & \mathbf{B}_2 \\ \mathbf{B}_2 & \mathbf{B}_1 \\ \mathbf{B}_0 & \mathbf{B}_0 \end{bmatrix}$$

in which

$$\mathbf{B}_1 = \frac{1}{4}\begin{bmatrix} 0 & 0 \\ -\frac{16}{5}b_3 & 0 \\ 0 & \frac{16}{5}b_2 \end{bmatrix}, \quad \mathbf{B}_2 = \frac{1}{4}\begin{bmatrix} 0 & 0 \\ \frac{16}{5}a_3 & 0 \\ 0 & -\frac{16}{5}a_2 \end{bmatrix}$$

$$\mathbf{B}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The following three points should be noted.

(1) For incompressible materials ($\nu = 0.5$), the internal parameter $\mathbf{h}^e$ cannot be eliminated through static condensation at the individual element stage, and must be dealt with only after assembly, through using techniques such as the front solution. When the front solution is used, it is best to eliminate the internal parameter λ first, the modal parameter second, and lastly the average compressive stress parameter $\mathbf{h}^e$.

(2) For nearly incompressible materials ($\nu \to 0.5$) or ordinary materials ($\nu < 0.5$), it is however possible to eliminate the parameters $\mathbf{h}^e$ and λ directly at the individual element stage through static condensation. Experience has shown that there is no deterioration in the accuracy for problems with $\nu = 0.49999999$ when the computation is carried out on an IBM-PC-AT computer in double precision.

(3) While stresses can be computed from $\boldsymbol{\sigma} = \mathbf{A}\boldsymbol{\varepsilon}$ or $\boldsymbol{\sigma} = \mathbf{P}\boldsymbol{\beta}$, the most accurate method is the one based on $\boldsymbol{\sigma}_1 = \mathbf{P}_1\boldsymbol{\beta}_1$. This is particularly true for highly distorted elements by Chen and Cheung (1987a,b) and Cheung and Chen (1988).

NUMERICAL EXAMPLES

Two examples will be presented to demonstrate the accuracy and versatility of the proposed hybrid elements when applied to incompressible or nearly incompressible material problems. All units are in kg and cm.

Incompressible material problem

The first example concerns a square block under uniform compression. The dimensions and other details are given in Fig. 2.

Only one Q_{4-LL} or Q_{4-C} element is used for the analysis. Since the Poisson's ratio is taken as 0.5, the computed displacement turns out to be, as expected, zero. The average compressive stress is given as $\sigma_p = 150$, which corresponds to the exact solution. However, in case element Q_{4-L} is used for the analysis, then as a result of the locking phenomenon, the value of the average compressive stress approaches infinity.

Compressible and nearly incompressible material problem

A cantilever beam under a shear load at the free end is analysed for a wide range of Poisson's ratios. The dimensions and other details are given in Fig. 3.

The computed vertical deflection at A and direct stress at B (Fig. 3) are listed in Tables 3 and 4, respectively. The computed deflection results are also presented diagramatically in Fig. 4. From the results it can be seen that both Q_{4-LL} and Q_{4-C} will not exhibit any locking phenomenon, and that Q_{4-LL} will yield more accurate results, particularly for stresses. For nearly incompressible material problems Q_{4-L} is not a suitable element, since locking will invariably occur as $\nu \rightarrow 0.5$.

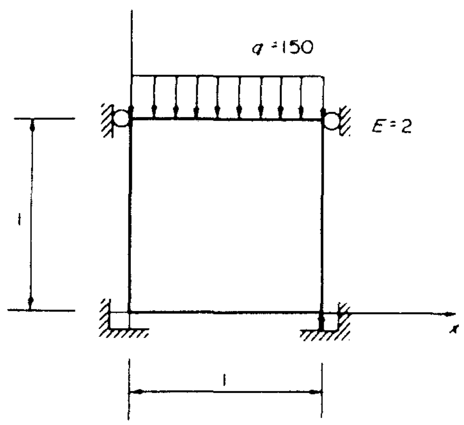

Fig. 2. Square block under uniform compression.

Table 3. Vertical deflection at point A of cantilever beam for different Poisson's ratios

	Elements									
	Q_{4-LL}			Q_{4-C}			Q_{4-L}			
	Results									
ν	1 × 1	1 × 5	2 × 10	1 × 1	1 × 5	2 × 10	1 × 1	1 × 5	2 × 10	Exact
0.3	1062.750	1390.350	1409.70	147.30	1326.020	1392.74	141.6	897.0	1237.4	1406.925
0.49	899.567	1173.152	1196.496	168.866	1519.800	1268.920	88.8	129.8	326.9	1190.026
0.4999	888.859	1158.895	1182.841	169.989	1529.898	1258.787	45.7	45.9	48.6	1175.771
0.49999	888.761	1158.765	1182.284	169.999	1529.990	1258.693	45.1	45.1	45.4	1175.640
0.499999	888.751	1158.751	1182.271	170.00	1529.999	1258.684	45.0	45.0	45.0	1175.626
0.4999999	888.750	1158.750	1182.270	170.00	1530.000	1258.680	45.0	45.0	45.0	1175.625
0.49999999	888.749	1158.750	1182.270	170.00	1530.00	1258.689	45.0	45.0	45.0	1175.625

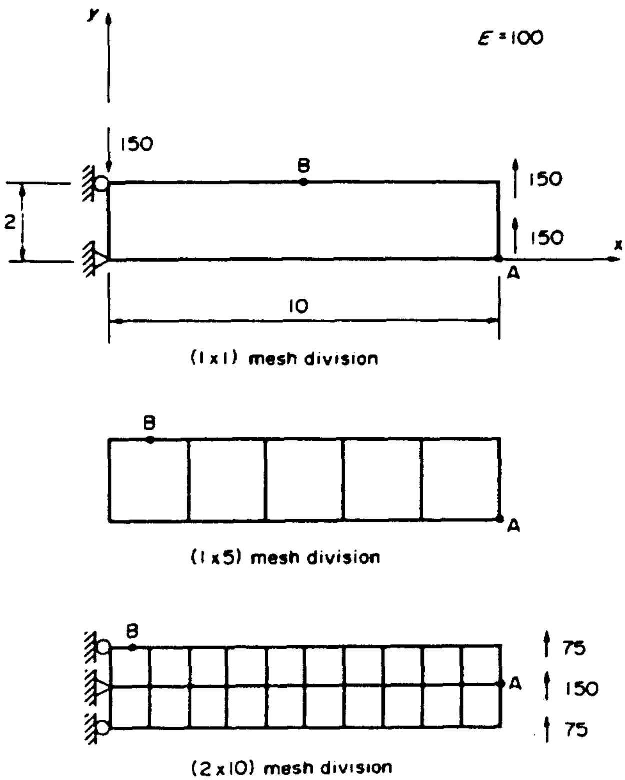

Fig. 3. Cantilever beam under shear load at free end.

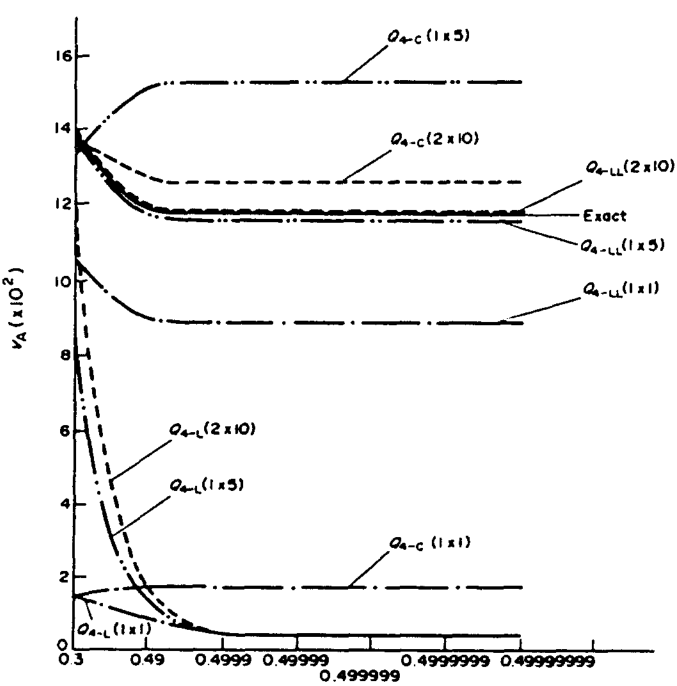

Fig. 4. A comparison of the computed v_A results for the cantilever beam with different Poisson's ratios.

Table 4. Direct stress σ_x at point B of cantilever beam for different Poisson's ratios

	Elements								
	Q_{4-LL}			Q_{4-C}			Q_{4-L}		
	Results								
ν	1 × 1	1 × 5	2 × 10	1 × 1	1 × 5	2 × 10	1 × 1	1 × 5	2 × 10
0.3	−2250.0	−4050.0	−4293.3	−166.7	−2700.0	−3598.3	−276.2	−3150.0	−4173.0
0.49	−2250.0	−4050.0	−4311.3	−166.7	−2700.0	−3445.3	−1509.9	−3972.1	−7388.4
0.4999	−2250.0	−4050.0	−4312.5	−166.7	−2700.0	−3436.7	−2238.8	−4049.2	−8535.0
0.49999	−2250.0	−4050.0	−4312.5	−166.7	−2700.0	−3436.6	−2248.8	−4049.9	−8548.5
0.499999	−2250.0	−4049.8	−4312.3	−166.7	−2700.2	−3436.7	−2249.9	−4050.1	−8549.8
0.4999999	−2250.0	−4048.1	−4316.0	−166.7	−2701.7	−3440.1	−2249.7	−4050.0	−8549.6
0.49999999	−2249.7	−4048.6	−4269.55	−166.7	−2684.5	−3383.1	−2251.0	−4047.0	−8550.6
Exact	−2250.0	−4050.0	−4275.0	−2250.0	−4050.0	−4275.0	−2250.0	−4050.0	−4275.0

CONCLUSIONS

(1) In order to develop hybrid element models which can be applied to problems with various Poisson's ratios (including incompressible and nearly incompressible materials), it is necessary first of all to establish a new variational functional π_G. Additional variables have been introduced into the finite element formulation for the purpose of efficiency and versatility. The introduction of H_c and H_0 facilitates the derivation of equations. The parameter σ_1 is used to improve the computational accuracy of stresses, while σ will help to increase the non-conforming internal displacement parameter u_λ effectively, thus avoiding the matrix inversion operations in stiffness formulation required by other techniques.

(2) The element Q_{4-LL} with three average compressive stress parameters and four internal displacement parameters, is an excellent element for incompressible and nearly incompressible material problems and is capable of yielding accurate displacement and stress results.

(3) The element Q_{4-C} with one average compressive stress parameter can be applied to incompressible and nearly incompressible material problems, and will not exhibit any singular or locking phenomenon. However, the accuracy of the results is inferior to that of Q_{4-LL}. It can be shown that for rectangular mesh divisions Q_{4-C} will produce the same results as the element developed by Hughes (1977).

(4) The element Q_{4-L} with three average compressible stress parameters is not a suitable element for incompressible and nearly incompressible material problems. It can be shown that for rectangular mesh divisions, Q_{4-L} will produce the same results as the conforming displacement Q_4.

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